

$N=2$ Electric-magnetic duality in a chiral background[†]

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N=2 Electric-magnetic duality in a chiral background

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We establish the consistency of duality transformations for generic systems of $N = 2$ vector supermultiplets in the presence of a chiral background field. This is relevant, for instance, when dealing with spurion fields or when considering higher-derivative couplings of vector multiplets to supergravity. We point out that under duality most quantities do not transform as functions. With few exceptions, true functions are nonholomorphic, even though the duality transformations themselves are holomorphic in nature.

1. Introduction

As is well known systems of abelian $N = 2$ vector supermultiplets transform systematically under duality transformations: transformations that act on the (abelian) field strengths and rotate the combined field equations and Bianchi identities by means of a symplectic matrix. This was first exploited for pure $N = 2$ supergravity [1]. For generic $N = 2$ vector supermultiplets it was discovered [2] that these transformations rotate the scalar fields X^I and the derivatives F_I of the holomorphic function $F(X)$ that encodes the Lagrangian, by means of an $Sp(2n+2; \mathbf{R})$ transformation, where n denotes the number of vector multiplets¹. Initially the emphasis was on invariances of the equations of motion. The fact that the scalars in supergravity often parametrize an homogeneous space whose transitive isometries are realized by duality transformations, enables one to conveniently control the nonpolynomial dependence on the scalar fields. Later it was realized that these transformations can also be used to reparametrize the theory in terms of a different function $\tilde{F}(\tilde{X})$ [3]. For the subgroup of the symplectic group corresponding to an invariance of the equations of motion, the function remains the same.

More recently symplectic reparametrizations

were exploited by Seiberg and Witten [4] and later by others [5], in obtaining exact solutions of low-energy effective actions for $N = 2$ supersymmetric Yang-Mills theory. The singularities in these effective actions signal their breakdown due to the emergence of massless states corresponding to monopoles and dyons. Although these states are the result of nonperturbative dynamics, they are nevertheless accessible because at these points one conveniently converts to an alternative dual formulation, in which local field theory is again applicable. In many of the nonperturbative solutions the quantities (X^I, F_J) can be defined as the periods of a meromorphic differential corresponding to a class of hyperelliptic curves. A similar phenomenon is known from type-II string compactifications on Calabi-Yau manifolds, where (X^I, F_J) can be associated with the periods of the $(3, 0)$ form of the Calabi-Yau three-fold.

2. $N = 2$ Vector multiplets

The actions we use are based on $N = 2$ chiral superspace integrals,

$$S \propto \text{Im} \left(\int d^4x d^4\theta F(W^I) \right), \quad (1)$$

where F is an arbitrary function of reduced chiral multiplets $W^I(x, \theta)$. Such multiplets carry the gauge-covariant degrees of freedom of a vector multiplet, consisting of a complex scalar X^I , a spinor doublet Ω^{iI} , a selfdual field-strength $F_{\mu\nu}^{-I}$ and a triplet of auxiliary fields Y_{ij}^I . This

¹Not counting the graviphoton of $N = 2$ supergravity. In the rigid case the symplectic matrix is only $2n$ -dimensional. Nonperturbatively, the symplectic transformations are usually restricted to a discrete subgroup.

Lagrangian may coincide with the effective Lagrangian associated with some supersymmetric Yang-Mills theory, but for our purposes its origin is not directly relevant. To enable coupling to supergravity the holomorphic function should be homogeneous of second degree.

The Lagrangian contains spin-1 kinetic terms proportional to

$$\mathcal{L} \propto i \left(\mathcal{N}_{IJ} F_{\mu\nu}^{+I} F^{+\mu\nu J} - \bar{\mathcal{N}}_{IJ} F_{\mu\nu}^{-I} F^{-\mu\nu J} \right), \quad (2)$$

where $F_{\mu\nu}^{\pm I}$ are the (anti-)selfdual field strengths and \mathcal{N} is proportional to the second derivative of the function $\tilde{F}(\tilde{X})$. In addition there are moment couplings (to the fermions, or to certain background fields, to be discussed later), so that the field strengths $F_{\mu\nu}^{\pm I}$ couple linearly to tensors $\mathcal{O}_I^{\pm\mu\nu}$, whose form is left unspecified at the moment. Defining

$$G_{\mu\nu I}^+ = \mathcal{N}_{IJ} F_{\mu\nu}^{+J} + \mathcal{O}_{\mu\nu I}^+, \quad (3)$$

and the corresponding anti-selfdual tensor that follows from complex conjugation, the Bianchi identities and equations of motion for the Abelian gauge fields take the form

$$\partial^\mu (F^+ - F^-)_{\mu\nu}^I = \partial^\mu (G^+ - G^-)_{\mu\nu I} = 0. \quad (4)$$

These are invariant under the transformation

$$\begin{pmatrix} F_{\mu\nu}^{\pm I} \\ G_{\mu\nu I}^{\pm} \end{pmatrix} \longrightarrow \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} F_{\mu\nu}^{\pm I} \\ G_{\mu\nu I}^{\pm} \end{pmatrix}, \quad (5)$$

where U_J^I , V_I^J , W_{IJ} and Z^{IJ} are constant real $(n+1) \times (n+1)$ submatrices.

From (3) and (5) one derives that \mathcal{N} must transform as

$$\tilde{\mathcal{N}}_{IJ} = (V_I^K \mathcal{N}_{KL} + W_{IL}) [(U + Z\mathcal{N})^{-1}]^L_J. \quad (6)$$

To ensure that \mathcal{N} remains a symmetric tensor, at least in the generic case, the transformation (5) must be an element of $Sp(2n+2, \mathbf{R})$ (we disregard a uniform scale transformation). Furthermore the tensor \mathcal{O} must change according to

$$\tilde{\mathcal{O}}_{\mu\nu I}^+ = \mathcal{O}_{\mu\nu J}^+ [(U + Z\mathcal{N})^{-1}]^J_I. \quad (7)$$

The required change of \mathcal{N} is induced by a change of the scalar fields, implied by

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix}. \quad (8)$$

In this transformation we include a change of F_I . Because the transformation is symplectic, one can show that the new quantities \tilde{F}_I can be written as the derivatives of a new function $\tilde{F}(\tilde{X})$. The new but equivalent set of equations of motion one obtains by means of the symplectic transformation (properly extended to other fields), follows from the Lagrangian based on \tilde{F} .

It is possible to integrate (8) and one finds

$$\begin{aligned} \tilde{F}(\tilde{X}) = & F(X) - \frac{1}{2} X^I F_I(X) \\ & + \frac{1}{2} (U^T W)_{IJ} X^I X^J \\ & + \frac{1}{2} (U^T V + W^T Z)_I{}^J X^I F_J \\ & + \frac{1}{2} (Z^T V)^{IJ} F_I F_J, \end{aligned} \quad (9)$$

up to a constant and terms linear in the \tilde{X}^I . In the coupling to supergravity, where the function must be homogeneous of second degree, such terms are obviously excluded².

The above expression (9) is not so useful, as it requires substituting \tilde{X}^I in terms of X^I , or vice versa. When F remains unchanged, $\tilde{F}(\tilde{X}) = F(\tilde{X})$, the theory is *invariant* under the corresponding transformations, but again this is hard to verify explicitly in this form. A more convenient method instead, is to verify that the substitution $X^I \rightarrow \tilde{X}^I$ into the derivatives $F_I(X)$ correctly induces the symplectic transformations on the periods (X^I, F_I) .

The result (9) shows immediately that

$$F(X) - \frac{1}{2} X^I F_I(X) \quad (10)$$

transforms as a function under the symplectic transformations, i.e., as $\tilde{f}(\tilde{X}) = f(X)$, something that is obviously not true for the quantity $F(X)$. Here we should stress that, although we generically denote quantities such as $F(X)$ as holomorphic functions, we now introduce an important distinction by insisting that certain quantities transform as functions *under symplectic transformations*. As we shall see, quantities with this property are more rare.

²The terms linear in \tilde{X} in (9) are associated with constant translations in \tilde{F}_I in addition to the symplectic rotation shown in (8). Likewise one may introduce constant shifts in \tilde{X}^I . Henceforth we ignore these shifts, which are excluded for local supersymmetry, even in the presence of a background. Constant contributions to $F(X)$ are always irrelevant.

In the coupling to supergravity, the above statement has no content, as (10) vanishes identically by virtue of the homogeneity of $F(X)$. In other situations, however, the fact that (10) transforms as a function under symplectic transformations is relevant. For example, the result has appeared in the literature in the context of the effective action of supersymmetric Yang-Mills theories, where it forms an ingredient in proving that (10) must in fact be invariant under a certain subgroup of the symplectic transformations. To see how this goes, we first note that

$$\delta(2F(X) - X^I F_I(X)) = (\delta X^I) F_I - X^I (\delta F_I), \quad (11)$$

under arbitrary variations. In particular we may consider changes of the moduli u_α that parametrize the ground states of the theory and identify δ in the above equation with derivatives with respect to these moduli. When the (X^I, F_I) can be defined in terms of the periods of a certain differential on a Riemann surface [4], whose moduli space is that of the Yang-Mills theory, they are subject to Picard-Fuchs equations. The latter are partial differential equations involving multiple derivatives of the periods with respect to the u_α . Combining these Picard-Fuchs equations with (11), one can show that $2F(X) - X^I F_I(X)$ satisfies a similar differential equation, which restricts it to a certain polynomial of the u_α . For the details of this, we refer to [6–8].

What we like to stress here is the following. Knowing that (10) can somehow be expressed in terms of the moduli, we conclude that it will not change when applying symplectic transformations belonging to the monodromy group. Hence, using that (10) transforms as a *function* under symplectic transformations, we derive that it must be a function that is invariant under the monodromy subgroup.

3. Backgrounds

We now reconsider supersymmetric Yang-Mills theory in the presence of a chiral background field and a conformal supergravity background. To couple supersymmetric vector multiplets to (scalar) chiral background fields is straightforward. One simply incorporates additional chiral

fields Φ into the function F that appears in the integrand of (1),

$$S \propto \text{Im} \left(\int d^4x d^4\theta F(W^I, \Phi) \right). \quad (12)$$

Also the coupling to conformal supergravity is known [9]. We draw attention to the fact that the W^I are reduced, while the Φ can be either reduced or general chiral fields.

Let us briefly discuss a few situations where such chiral backgrounds are relevant; in the next section we turn to explicit formulae. In supersymmetric theories many of the parameters (coupling constants, masses) can be regarded as background fields that are frozen to constant values (so that supersymmetry is left intact). Because these background fields correspond to certain representations of supersymmetry, the way in which they appear in the theory – usually both perturbatively as well as nonperturbatively – is restricted by supersymmetry. In this way we may derive restrictions on the way in which parameters can appear. An example is, for instance, the coupling constant and θ angle of a supersymmetric gauge theory, which can be regarded as a chiral field frozen to a complex constant $iS = \theta/2\pi + i4\pi/g^2$. Supersymmetry now requires that the function $F(X)$ depends on S , but *not* on its complex conjugate. This strategy of introducing so-called *spurion* fields is not new. In the context of supersymmetry it has been used in, for instance, [10–12] to derive nonrenormalization theorems and even exact results.

Spurion fields can also be used for mass terms of hypermultiplets. When considering the effective action after integrating out the hypermultiplets, the dependence on these mass parameters can be incorporated in chiral background fields. In this example the background must be restricted to reduced chiral multiplets. In the previous example this restriction is optional. On the other hand, it may also be advantageous to not restrict the background fields to constant values, in order to introduce an explicit breaking of supersymmetry [13,14].

Another context where chiral backgrounds are relevant concerns the coupling to the Weyl multiplet, which involves interactions of vector multi-

plets to the square of the Riemann tensor. In this case the scalar chiral background is not reduced and is proportional to the square of the Weyl multiplet. Here the strategy is not, of course, to freeze the background to a constant, but one is interested in more general couplings with conformal supergravity. We return to a more detailed discussion of the Weyl multiplet shortly.

We add that all of this is very natural from the point of view of string theory, where the moduli fields, which characterize the parameters of the (supersymmetric) low-energy physics, reside in supermultiplets. In heterotic $N = 2$ compactifications the background field S introduced above coincides with the complex dilaton field, which comprises the dilaton and the axion, and belongs to a vector multiplet. The dilaton acts as the loop-counting parameter for string perturbation theory. Although the full supermultiplet that contains the dilaton is now physical, the derivation of nonrenormalization theorems can proceed in the same way [15,16]. We should stress here that when restricting the background to a reduced chiral multiplet, one can just treat it as an additional (albeit external) vector multiplet. Under these circumstances one may consider extensions of the symplectic transformations that involve also the background itself. Of course, when freezing the background to constant values, one must restrict the symplectic transformations accordingly. The above strategy is especially useful when dealing with anomalous symmetries. By extending anomalous transformations to the background fields, the variation of these fields can compensate for the anomaly. The extended non-anomalous symmetry becomes again anomalous once the background is frozen to a constant value. This strategy is particularly valuable when dealing with massive hypermultiplets.

In the second half of this section we discuss a number of features pertaining to the Weyl multiplet [17]. The bosonic fields of $N = 2$ conformal supergravity consist of the vierbein field e_μ^a , $SU(2) \times U(1)$ chiral gauge fields A_μ and $V_{\mu j}^i$ (associated with the automorphism group of the supersymmetry algebra), a selfdual tensor field T_{abij} , antisymmetric in both Lorentz and $SU(2)$ indices, and a scalar field D . The fermionic fields,

which we will mostly ignore here, are the gravitino fields ψ_μ^i and a spinor field χ^i . In coupling the chiral Lagrangian (12) to conformal supergravity, all of these fields will appear. However, the covariant quantities associated with this field representation form themselves a chiral selfdual tensor multiplet. Its lowest-weight component is the antisymmetric tensor T_{ab}^{ij} , the next one consists of χ^i and the field-strength of the gravitino field (with modifications such that it is superconformally covariant); then, at level θ^2 we have the selfdual component of the Riemann tensor and the $SU(2) \times U(1)$ field strengths (all of them with proper superconformal modifications) as well as D and $T_{abij}T_{cd}^{ij}$. All higher- θ components are dependent, in view of the fact that we are dealing with a reduced chiral multiplet.

The square of the Weyl multiplet constitutes a scalar chiral field of scaling weight 2. Its lowest component is equal to $(\varepsilon_{ij}T_{ab}^{ij})^2$; the highest- θ component contains the square of the selfdual components of the Riemann tensor and the chiral field strengths, as well as a variety of other terms. We refer to [17] for details.

By associating the chiral background field Φ with the square of the Weyl multiplet \mathcal{W} , we thus obtain new (higher-derivative) couplings of vector multiplets with conformal supergravity. Assume that the function F can be expanded as a power series,

$$F(X, \mathcal{W}^2) = \sum_{g=0}^{\infty} F^{(g)}(X) (\mathcal{W}^2)^g. \quad (13)$$

Because it must be homogeneous of second degree with scaling weights of X and \mathcal{W} that are both equal to unity, the coefficient functions $F^{(g)}(X)$ are homogeneous of degree $2(1 - g)$.

In supergravity the X^I are not independent scalar fields, but are defined projectively; in more mathematical terms they can be regarded as sections of a complex line bundle. These sections can be expressed holomorphically in terms of independent complex fields z^A , which describe the physical scalars of the vector multiplets. The original quantities X^I and the holomorphic sections $X^I(z)$ differ by a factor $m_P \exp(K/2)$, where K

is the Kähler potential³ and m_P is the Planck mass. In view of the projective nature of the X^I , there is thus always one more physical vector field than there are physical scalars. The extra vector corresponds to the graviphoton. The Lagrangian encoded by (13) gives rise to terms proportional to the square of the Riemann tensor times $(\varepsilon_{ij} T_{ab}^{ij})^{2(g-1)}$. After extracting the scale factor $m_P \exp(K/2)$, the coefficient functions $F^{(g)}(X)$ give rise to holomorphic functions $\mathcal{F}^{(g)}$ of the z (or rather sections of a line bundle).

Let us consider the case where the function (13) encodes the $N = 2$ supersymmetric effective low-energy field theory corresponding to a type-II string compactification on a Calabi-Yau manifold. The Planck mass m_P is equal to the string scale divided by the string coupling constant g_S ; the latter is proportional to the dilaton. In a type-II string compactification the dilaton, which counts string loops, does not reside in the vector multiplet sector, so that the prefactor $\exp(K/2)$ is independent of the string coupling constant and the X^I are proportional to g_S^{-1} . Consequently $(\mathcal{W})^g$ is multiplied by terms of order $g_S^{2(g-1)}$, which thus represent g -loop contributions in string perturbation theory. The coefficient functions can be determined in string theory from certain type-II string amplitudes [18] and indeed arise in the appropriate orders in string perturbation theory. An interesting feature is that the $\mathcal{F}^{(g)}$ can be identified with the topological partition function of a twisted nonlinear sigma model on a Calabi-Yau target space, defined on a two-dimensional base space equal to a genus- g Riemann surface. The partition function is obtained by integrating appropriately over all these Riemann surfaces [19]. However, the partition functions $\mathcal{F}^{(g)}$ do not depend holomorphically on the Calabi-Yau moduli. They exhibit a so-called

³In terms of the holomorphic sections the Kähler potential takes the form

$$K(z, \bar{z}) = -\log \left(i \bar{X}^I(\bar{z}) F_I(X(z)) - i X^I(z) \bar{F}_I(\bar{X}(\bar{z})) \right).$$

The Kähler metric is defined as $g_{A\bar{B}} = \partial_A \partial_{\bar{B}} K(z, \bar{z})$. Under projective transformations of the holomorphic sections, $X^I \rightarrow \exp(f(z)) X^I$, the Kähler potential transforms by a Kähler transformation, so that the metric remains invariant.

holomorphic anomaly due to the propagation of massless states, or equivalently, due to certain contributions from the boundary of the moduli space \mathcal{M}_g associated with the genus- g Riemann surfaces. The holomorphic anomaly is governed by the following equations (with certain normalizations of the $\mathcal{F}^{(g)}$) [19],

$$\begin{aligned} \partial_{\bar{A}} \mathcal{F}^{(g)} &= \frac{1}{2} e^{2K} \bar{\mathcal{W}}_{\bar{A}}{}^{BC} \\ &\times \left[D_B D_C \mathcal{F}^{(g-1)} + \sum_{r=1}^{g-1} D_B \mathcal{F}^{(r)} D_C \mathcal{F}^{(g-r)} \right], \end{aligned} \quad (14)$$

for $g > 1$, whereas for $g = 1$ we have

$$\begin{aligned} \partial_A \partial_{\bar{B}} \mathcal{F}^{(1)} &= \frac{1}{2} e^{2K} \mathcal{W}_{ACD} \bar{\mathcal{W}}_B{}^{CD} - \frac{1}{24} \chi g_{AB} \\ &= -\frac{1}{2} R_{AB} + \frac{1}{2} (n+1 - \frac{1}{12} \chi) g_{AB}, \end{aligned} \quad (15)$$

with $R_{A\bar{B}}$ the Ricci tensor and χ the Euler number of the Calabi-Yau moduli space.⁴ In these equations target-space indices are raised or lowered by means of the Kähler metric. Covariant derivatives are projectively covariant and defined by

$$D_A \mathcal{F}^{(g)} = (\partial_A + 2(1-g) \partial_A K) \mathcal{F}^{(g)}, \quad (16)$$

and furthermore we used the definition

$$\begin{aligned} \mathcal{W}_{ABC} &= i F_{IJK}(X(z)) \\ &\times \frac{\partial X^I(z)}{\partial z^A} \frac{\partial X^J(z)}{\partial z^B} \frac{\partial X^K(z)}{\partial z^C}. \end{aligned} \quad (17)$$

In $N = 2$ compactifications of the heterotic string the counting of string loops runs differently, because here the Kähler potential depends explicitly on the dilaton field. Now the dilaton dependence in $\exp(K/2)$ cancels the string coupling constant induced by the Planck mass, so that the X^I are generically of order zero in the string coupling constant. However, the dilaton coincides with one of the fields z (or some function of them). The actual dependence on the string coupling constant is therefore directly governed by the explicit dilaton dependence of the

⁴Here we used that the moduli space is an n -dimensional special Kähler space. A particular solution of (15) is $\mathcal{F}^{(1)} = -\frac{1}{2} \ln g + \frac{1}{2} (n+1 - \frac{1}{12} \chi) K$, where g is the metric determinant.

quantities $\mathcal{F}^{(g)}$. Interestingly enough, this explicit dependence is restricted, at least in perturbation theory, by virtue of a nonrenormalization theorem. Based on this theorem, in heterotic compactifications, one expects the $\mathcal{F}^{(g)}$ to appear at the one-loop level, with the exception of $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$ which also receive classical contributions [20]. Beyond this there are of course nonperturbative terms.

The above observations are relevant in certain explicit tests of ‘string duality’ between heterotic string compactifications on $K_3 \times T_2$ and type-II string compactifications on Calabi-Yau manifolds [21]. In these tests [20,22,23] the nonperturbative effects on the Calabi-Yau side are compared to the perturbative effects on the heterotic side. The latter were studied in [16,25].

4. Duality in a chiral background

After this somewhat qualitative discussion let us turn to more explicit results. In this section we verify the consistency of symplectic reparametrizations in a general chiral background. From this we learn how coefficient functions such as in (13) change under these reparametrizations. As it turns out, they do not, in general, transform as functions, as is already suggested by the transformation rule (9) for $F(X)$.

The subsequent discussion is based on the action (12). A first observation is that, a priori, it is not meaningful to restrict the dependence on the background field. For instance, one may couple the theory linearly to the background, so that \mathcal{N} will depend at most linearly on the background field. However, after a symplectic transformation, \mathcal{N} will generically have a nonlinear dependence on the background, as follows from (6). Therefore the only meaningful approach is to start from functions F which depend both on the gauge superfield strengths W^I and on the background field Φ in a way that is a priori unrestricted. Then one can proceed exactly as before and examine the equivalence classes in the presence of the background. The transformation rules, however, will also depend on the background fields. This does not affect the derivation, although there are a

number of new features.

We will consider the component expression corresponding to (12) ignoring the fermions. We assume the presence of one chiral scalar background (the generalization to more background fields is straightforward), which itself may be equal to a scalar expression of nonscalar chiral fields as in (13). We denote the bosonic components of the chiral background superfield by \hat{A} , \hat{B}_{ij} , \hat{F}_{ab}^- and \hat{C} . Here \hat{A} and \hat{C} are complex scalars, appearing at the θ^0 and θ^4 level of the chiral superfield, while the symmetric complex $SU(2)$ tensor \hat{B}_{ij} and the anti-selfdual Lorentz tensor \hat{F}_{ab}^- reside at the θ^2 level. The holomorphic function F now depends on the lowest- θ components X and \hat{A} of the gauge superfield strengths and the background field, respectively. The supersymmetric Lagrangian is proportional to the *real* part of the following expression

$$\begin{aligned} & -2iF_I(\partial_\mu - iA_\mu)^2 \bar{X}^I \\ & -\frac{1}{4}iF_{IJ}Y_{ij}^I Y^{Jij} - \frac{1}{2}i\hat{B}_{ij}F_{AI}Y^{Iij} \\ & +\frac{1}{2}iF_{IJ}(F_{ab}^{-I} - \frac{1}{4}\bar{X}^I T_{ab}^{ij}\varepsilon_{ij})(F_{ab}^{-J} - \frac{1}{4}\bar{X}^J T_{ab}^{ij}\varepsilon_{ij}) \\ & -\frac{1}{4}iF_I(F_{ab}^{+I} - \frac{1}{4}X^I T_{abij}\varepsilon^{ij})T_{abij}\varepsilon^{ij} \\ & -i\hat{F}_{ab}^-(F_{AI}F_{ab}^{-I} - \frac{1}{4}F_{AI}\bar{X}^I T_{ab}^{ij}\varepsilon_{ij}) \\ & +iF_A\hat{C} - \frac{1}{4}iF_{AA}(\varepsilon^{ik}\varepsilon^{jl}\hat{B}_{ij}\hat{B}_{kl} - 2\hat{F}_{ab}^-\hat{F}_{ab}^-) \\ & -\frac{1}{16}iF(T_{abij}\varepsilon^{ij})^2 - 2i(\frac{1}{6}R - D)F_I\bar{X}^I, \end{aligned} \quad (18)$$

where we also included the coupling to the bosonic fields of conformal supergravity. However, the vierbein determinant has been suppressed, while the $SU(2)$ gauge fields do not appear because they couple only to fermions. Note that (18) depends only on derivatives of the function $F(X, \hat{A})$ with respect to the X^I and/or the background field \hat{A} . In the notation used previously we immediately derive the following expressions from (18),⁵

$$\mathcal{N}_{IJ} = \bar{F}_{IJ},$$

⁵Note that the expression for \mathcal{N} takes this form irrespective of whether we couple to supergravity. This is so because the auxiliary tensor T has not been integrated out; therefore we must insist that the function F exists. After integrating out the auxiliary tensor it is possible to reformulate the theory in such a way that the function F no longer needs to exist, as long as the periods (X^I, F_J) can consistently be written down [24].

$$\mathcal{O}_{\mu\nu I}^+ = \frac{1}{4}(F_I - \bar{F}_{IJ}X^J)T_{\mu\nu ij}\varepsilon^{ij} - \hat{F}_{\mu\nu}^+ \bar{F}_{IA}. \quad (19)$$

It is convenient to introduce the following definitions,

$$\begin{aligned} \frac{\partial \tilde{X}^I}{\partial X^J} &\equiv \mathcal{S}^I_J(X, \hat{A}) = U^I_J + Z^{IK} F_{KJ}, \\ \mathcal{Z}^{IJ} &\equiv [\mathcal{S}^{-1}]^I_K Z^{KJ}, \\ N_{IJ} &\equiv 2 \operatorname{Im} F_{IJ}, \quad N^{IJ} \equiv [N^{-1}]^{IJ}. \end{aligned} \quad (20)$$

The quantity \mathcal{Z}^{IJ} is symmetric in I and J , because ZU^T is a symmetric matrix as a consequence of the fact that U and Z are certain submatrices of a symplectic matrix.

The symplectic reparametrizations should act on the quantities (19) according to (6) and (7), which in the above notation read

$$\begin{aligned} \tilde{N}_{IJ} &= (V_I^K \mathcal{N}_{KL} + W_{IL}) [\bar{\mathcal{S}}^{-1}]^L_J, \\ \tilde{\mathcal{O}}_{\mu\nu I}^+ &= \mathcal{O}_{\mu\nu J}^+ [\bar{\mathcal{S}}^{-1}]^J_I. \end{aligned} \quad (21)$$

To ensure that these transformations are indeed realized, one introduces the transformations on the fields X^I , exactly as in section 2, except that the various quantities will now depend on the background field. So we have the transformation rule (8) and the same expression (9) for the new function after a symplectic transformation. Obviously the relation between X and \tilde{X} involves \hat{A} .

Irrespective of the background the quantity \mathcal{N} still transforms according to the first equation of (21). Also the following result,

$$\begin{aligned} \tilde{F}(\tilde{X}, \hat{A}) - \frac{1}{2} \tilde{X}^I \tilde{F}_I(\tilde{X}, \hat{A}) = \\ F(X, \hat{A}) - \frac{1}{2} X^I F_I(X, \hat{A}), \end{aligned} \quad (22)$$

still holds, so that there is a holomorphic function that transforms as a function under symplectic transformations. In the coupling to supergravity this result is still relevant, provided the background field \hat{A} has a nonzero scaling weight. Other results which hold irrespective of the background, are

$$\begin{aligned} \tilde{N}_{IJ} &= N_{KL} [\bar{\mathcal{S}}^{-1}]^K_I [\mathcal{S}^{-1}]^L_J, \\ \tilde{N}^{IJ} &= N^{KL} \bar{\mathcal{S}}^I_K \mathcal{S}^J_L, \\ \tilde{F}_{IJK} &= F_{MNP} [\mathcal{S}^{-1}]^M_I [\mathcal{S}^{-1}]^N_J [\mathcal{S}^{-1}]^P_K, \end{aligned} \quad (23)$$

where all quantities depend on both the fields X and \hat{A} . The symmetry of the first two quantities is preserved owing to the symplectic nature of the transformation.

Results that specifically refer to the background are obtained by taking derivatives of \tilde{F} (cf. (9)), keeping \tilde{X} fixed in partial differentiations of \tilde{F} with respect to \hat{A} , and/or using already known transformations. In this way we obtain, for instance,

$$\begin{aligned} \tilde{F}_A(\tilde{X}, \hat{A}) &= F_A(X, \hat{A}), \\ \tilde{F}_{AI} &= F_{AJ} [\mathcal{S}^{-1}]^J_I, \\ \tilde{F}_I - \tilde{F}_{IJ} \tilde{X}^J &= [F_J - F_{JK} X^K] [\mathcal{S}^{-1}]^J_I, \\ \tilde{F}_I - \tilde{F}_{IJ} \tilde{X}^J &= [F_J - \bar{F}_{JK} X^K] [\bar{\mathcal{S}}^{-1}]^J_I, \\ \tilde{F}_{AA}(\tilde{X}, \hat{A}) &= F_{AA}(X, \hat{A}) \\ &\quad - F_{AI}(X, \hat{A}) F_{AJ}(X, \hat{A}) \mathcal{Z}^{IJ}. \end{aligned} \quad (24)$$

These relations suffice to show that the transformation behaviour of the tensors \mathcal{O}^\pm defined by (19) is indeed in accord with (21).

For later use we also list a few results involving higher derivatives of $F(X, \hat{A})$,

$$\begin{aligned} \tilde{F}_{AIJ} &= (F_{AKL} - F_{AM} \mathcal{Z}^{MN} F_{NKL}) \\ &\quad \times [\mathcal{S}^{-1}]^K_I [\mathcal{S}^{-1}]^L_J, \\ \tilde{F}_{AAI} &= (F_{AAJ} - 2F_{AK} \mathcal{Z}^{KL} F_{ALJ} \\ &\quad + F_{JKL} (\mathcal{Z}^{KM} F_{AM}) (\mathcal{Z}^{LN} F_{AN})) [\mathcal{S}^{-1}]^J_I, \\ \tilde{F}_{AAA} &= F_{AAA} - 3F_{AAI} \mathcal{Z}^{IJ} F_{AJ} \\ &\quad + 3F_{AIJ} (\mathcal{Z}^{IK} F_{AK}) (\mathcal{Z}^{JL} F_{AL}) \\ &\quad - F_{IJK} (\mathcal{Z}^{IL} F_{AL}) (\mathcal{Z}^{JM} F_{AM}) (\mathcal{Z}^{KN} F_{AN}). \end{aligned} \quad (25)$$

What remains to show is that the full equations of motion and the Bianchi identities are equivalent under symplectic reparametrizations. This can be done by identifying the terms in the Lagrangian that vanish by virtue of the equations of motion for the auxiliary fields and the vector fields. For the remaining terms one must then prove that they preserve their form under symplectic transformations. To do this we write the Lagrangian as follows,

$$\begin{aligned} \mathcal{L} &= \frac{1}{4} N_{IJ} \mathcal{Y}_{ij}^I \mathcal{Y}^{ijJ} \\ &\quad - \frac{1}{2} i (\partial_\mu W_\nu^I - \partial_\nu W_\mu^I) (G^+ - G^-)_I^{\mu\nu} \\ &\quad - 2i F_I (\partial_\mu - i A_\mu)^2 \tilde{X}^I + \text{h.c.} \end{aligned}$$

$$\begin{aligned}
& +\frac{1}{2}\hat{B}_{ij}\hat{B}^{ij}N^{IJ}\bar{F}_{AI}F_{AJ} \\
& +\frac{1}{4}[\hat{B}_{ij}\hat{B}_{kl}\varepsilon^{ik}\varepsilon^{jl}-2\hat{F}_{ab}^-\hat{F}_{ab}^-] \\
& \quad \times [N^{IJ}F_{AI}F_{AJ}-iF_{AA}] + \text{h.c.} \quad (26) \\
& -\frac{1}{8}i(G_{abI}^-\bar{X}^I-F_{ab}^-\bar{F}_I)T_{ab}^{ij}\varepsilon_{ij} + \text{h.c.} \\
& +\frac{1}{2}\hat{F}_{ab}^-(F_{ab}^{-K}\bar{F}_{KI}-G_{abI}^-)N^{IJ}F_{AJ} + \text{h.c.} \\
& +iF_A\hat{C} + \text{h.c.} \\
& +\frac{1}{16}i(\bar{F}-\frac{1}{2}\bar{X}^I\bar{F}_I)(T_{ab}^{ij}\varepsilon_{ij})^2 + \text{h.c.} \\
& -\frac{1}{8}\hat{F}_{ab}^-T_{ab}^{ij}\varepsilon_{ij}(\bar{F}_{IJ}\bar{X}^J-\bar{F}_I)N^{IK}F_{AK} + \text{h.c.} \\
& -2i(\frac{1}{6}R-D)(\bar{X}^IF_I-\bar{F}_IX^I),
\end{aligned}$$

where we redefined the auxiliary fields Y_{ij}^I according to

$$\mathcal{Y}_{ij}^I = Y_{ij}^I - iN^{IJ}[\hat{B}_{ij}F_{AJ} - \varepsilon_{ik}\varepsilon_{jl}\hat{B}^{kl}\bar{F}_{AJ}].$$

The field equation of the auxiliary fields puts \mathcal{Y} to zero, so that we can drop the first term in (26). Likewise, the field equations for the vector fields converts the second term in (26) into a total divergence.

Using the identities (22-24) it is not difficult to show that all the terms in (26), with the exception of the first two, preserve their form under symplectic transformations.⁶ This concludes the proof that duality transformations are consistent in the presence of a chiral background with coupling to supergravity.

We close this section by pointing out that restricting the background to a constant (i.e., \hat{A} constant and all other background components vanishing) just gives the standard coupling of vector multiplets to $N = 2$ conformal supergravity. On the other hand, keeping also the fields \hat{B} and/or \hat{C} nonzero causes a breakdown of supersymmetry. This feature of the background field can be exploited when studying certain $N = 1$ supersymmetric, or even nonsupersymmetric, theories.

⁶Here we note a useful theorem [26]: from a symplectic vector, such as $(F_{\mu\nu}^{\pm I}, G_{\mu\nu}^{\pm J})$, one can construct a quantity $\mathcal{V}_I = (G_{\mu\nu I}^{\pm} - F_{IJ}F_{\mu\nu}^{\pm J})$ which transforms under symplectic transformations as $\tilde{\mathcal{V}}_I = \mathcal{V}_J[S^{-1}]^J_I$. The third and fourth equation of (24) can be derived directly on the basis of this theorem.

5. Symplectic functions and holomorphy

The above results show that certain expressions can be constructed from the function $F(X, \hat{A})$ and its derivatives that transform as *functions* under symplectic transformations. One obvious example is the holomorphic expression (10); another one is F_A , the first derivative of F with respect to the background. However, higher than first derivatives of F with respect to \hat{A} do not transform as functions under symplectic transformations. This means that the coefficient functions in an expansion such as (13) do not transform as symplectic functions, with the exception of $\mathcal{F}^{(1)}$. The transformation rules for these coefficient functions follow from (24) and (25) and their generalizations, putting the background field \hat{A} to zero.

This conclusion may be somewhat disturbing especially when considering symplectic transformations that constitute an invariance. In that situation we have $\tilde{F}(\tilde{X}, \hat{A}) = F(\tilde{X}, \hat{A})$. In spite of that, this does not imply that the coefficient functions (i.e. multiple derivatives with respect to the background) are invariant *functions* under the corresponding transformations. This should only be the case for the first one corresponding to F_A .

One may wonder whether there are modifications of the multiple- \hat{A} derivatives of F that do transform as functions under symplectic transformations. Such functions should be expected to arise when evaluating the coefficient functions directly on the basis of some underlying theory, such as string theory. These modifications seem possible in view of the fact that the combination

$$F_{AA} + iN^{IJ}F_{AI}F_{AJ},$$

which appears in (26), does indeed transform as a function under symplectic transformations. The latter was in fact required in order for the duality transformations to be consistent in the presence of the full chiral background, as shown in the previous section. Likewise, one may verify by explicit calculation that there is a generalization of the third derivative,

$$F_{AAA} + 3iN^{IJ}F_{AAI}F_{AJ}$$

$$\begin{aligned} & -3N^{IK}N^{JL}F_{AIJ}F_{AK}F_{AL} \\ & -iN^{IL}N^{JM}N^{KN}F_{IJK}F_{AL}F_{AM}F_{AN}, \end{aligned}$$

which also transforms as a symplectic function.

It turns out that these functions can be generated systematically. Assume that $G(X, \hat{A})$ transforms as a function under symplectic transformations. Then one readily proves that also $\mathcal{D}G(X, \hat{A})$ transforms as a symplectic function, where⁷

$$\mathcal{D} \equiv \frac{\partial}{\partial \hat{A}} + iF_{AI}N^{IJ}\frac{\partial}{\partial X^J}. \quad (27)$$

Consequently one can directly write down a hierarchy of functions which are modifications of multiple derivatives $F_{A\dots A}$,

$$F^{(n)}(X, \hat{A}) \equiv \frac{1}{n!}\mathcal{D}^{n-1}F_A(X, \hat{A}), \quad (28)$$

where we included a normalization factor. All the $F^{(n)}$ transform as functions under symplectic functions. However, except for the first one, they are not holomorphic. The lack of holomorphy is governed by the following equation ($n > 1$),

$$\frac{\partial F^{(n)}}{\partial \bar{X}^I} = \frac{1}{2}\bar{F}_I{}^{JK}\sum_{r=1}^{n-1}\frac{\partial F^{(r)}}{\partial X^J}\frac{\partial F^{(n-r)}}{\partial X^K}, \quad (29)$$

where $\bar{F}_I{}^{JK} = \bar{F}_{ILM}N^{LJ}N^{MK}$. Interestingly enough, this equation coincides with (15), except that the first term on the right-hand side of (15) is absent here. This is the term that arises from Riemann surfaces where a closed loop is pinched, which lowers the genus by one unit. The second term, which coincides with the one above, corresponds to pinchings that separate the Riemann surface into two disconnected surfaces [19].

We should stress that (29) was obtained in a very general context and applies to both rigid and local $N = 2$ supersymmetry. In the latter case we have to convert to holomorphic sections $X^I(z)$. This requires to set $\hat{A} = 0$ in (29). The holomorphic anomaly can thus be viewed as arising from a conflict between the requirements of holomorphy and of a proper behaviour under symplectic transformations. The fact that there is not an additional term in the anomaly equation, as in

⁷We note that \mathcal{D} and $N^{IJ}\partial_J$ commute.

(15), is not unrelated to the fact that $F^{(1)}$ is still holomorphic.

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